

BIOS 6312: Modern Regression Analysis

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Set 14: Bootstrap Methods

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2 The nonparametric bootstrap

Confidence intervals and inverting the test:

- Consider the following general quantity, which follows a familiar form:

$$S = \frac{\hat{\theta} - \theta}{\widehat{SE}(\hat{\theta})}$$

- When using this quantity to construct CIs, we often rely on two particular properties:
 - ▶ S is *pivotal* in large samples, meaning its asymptotic distribution does not depend upon θ .
 - ▶ S possesses a distribution that is approximately symmetric about zero in large samples.

Confidence intervals and inverting the test:

- Consider a coefficient, β , from a regression model:

$$\frac{\hat{\beta} - \beta}{\widehat{SE}(\hat{\beta})} \sim t_{df}.$$

- Note that the pivotal property is embedded above. Further,

$$\begin{aligned} t_{\alpha/2, df} &\leq \frac{\hat{\beta} - \beta}{\widehat{SE}(\hat{\beta})} \leq t_{1-\alpha/2, df} \\ \iff t_{\alpha/2, df} \widehat{SE}(\hat{\beta}) &\leq \hat{\beta} - \beta \leq t_{1-\alpha/2, df} \widehat{SE}(\hat{\beta}) \\ \iff -t_{1-\alpha/2, df} \widehat{SE}(\hat{\beta}) &\leq \beta - \hat{\beta} \leq -t_{\alpha/2, df} \widehat{SE}(\hat{\beta}) \\ \iff \hat{\beta} - t_{1-\alpha/2, df} \widehat{SE}(\hat{\beta}) &\leq \beta \leq \hat{\beta} + t_{\alpha/2, df} \widehat{SE}(\hat{\beta}) \end{aligned}$$

- From symmetry property, further derive the following:

$$\hat{\beta} - t_{1-\alpha/2, df} \widehat{SE}(\hat{\beta}) \leq \beta \leq \hat{\beta} + t_{1-\alpha/2, df} \widehat{SE}(\hat{\beta})$$

- These properties are the basis for forming symmetric CIs based on large sample theory.

Confidence intervals and inverting the test:

- When no such pivotal quantity exists, confidence intervals can be obtained by directly inverting the test.
- “Find all $\beta^{(0)}$ such that $H_0 : \beta = \beta^{(0)}$ cannot be rejected.”

Confidence intervals and inverting the test:

- In linear regression, an *exact* distribution for $\hat{\beta}$ based on the t -distribution depends upon normality of the errors.
- That distribution is approximately correct for large samples even if normality does not hold.
- In smaller samples, the nonparametric bootstrap can be used to obtain CIs that do not rely on large sample theory.

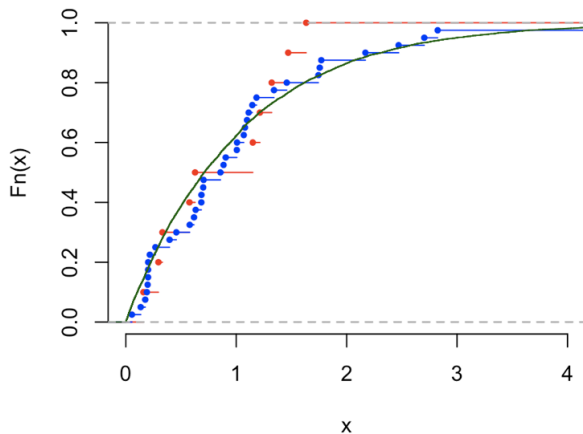
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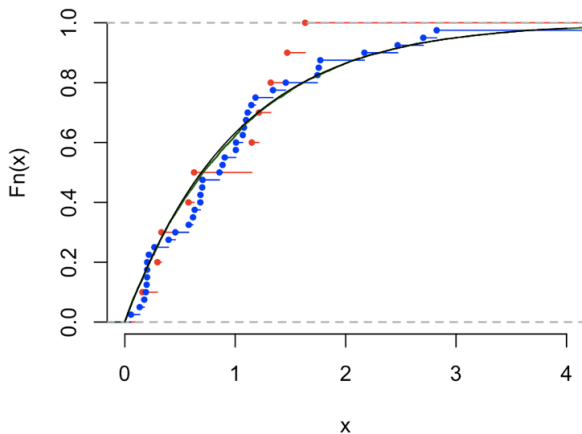
THE BOOTSTRAP

Preliminaries: \mathbb{F}_N , approximates $F(x) = P(X \leq x)$



THE BOOTSTRAP

Preliminaries: \mathbb{F}_N , approximates $F(x) = P(X \leq x)$



Main ideas:

- Let F denote cdf for (\mathbf{X}, Y) or $(Y|\mathbf{X})$, depending on context; let \mathbb{F}_N denote empirical cdf based on N observations.
 - ▶ $\boldsymbol{\beta} = T(F)$, and hence $\hat{\boldsymbol{\beta}} = T(\mathbb{F}_N)$.
 - ▶ Absent parametric form, \mathbb{F}_N is our best estimate of F .
- Repeat-sample of \mathbb{F}_N with replacement gives information on distribution of $\hat{\boldsymbol{\beta}}^* = T(\mathbb{F}_N^*)$; asterisk denotes fixed \mathbb{F}_N .
- Let $\{\hat{\boldsymbol{\beta}}_b^*\}_{b=1}^B$ denote the (bootstrap) samples.
- Note two layers of variation:
 - ▶ How well \mathbb{F}_N approximates F (better as $N \nearrow \infty$ by Glivencko-Cantelli: $\sup_{t \in [0,1]} |F(t) - \mathbb{F}_N(t)| \xrightarrow{\text{a.s.}} 0$).
 - ▶ How well $\{\hat{\boldsymbol{\beta}}_b^*\}_{b=1}^B$ approximates $T(\mathbb{F}_N^*)$ (better as $B \nearrow \infty$).
- Which source of variation can we better control?

Estimator-attributed bias:

- Let $\hat{\beta}_b^* = T(F_{N:b}^*)$ denote estimate based on b^{th} bootstrap sample. We may estimate bias as follows:

$$\begin{aligned}\widehat{\text{Bias}} &= \frac{1}{B} \sum_{b=1}^B (T(\mathbb{F}_{N:b}^*) - T(\mathbb{F}_N)) \\ &= \frac{1}{B} \sum_{b=1}^B \hat{\beta}_b^* - \hat{\beta} = \hat{\beta}^* - \hat{\beta} \approx \hat{\beta} - \beta.\end{aligned}$$

- Note that $\hat{\beta}^* = \frac{1}{B} \sum_{b=1}^B \hat{\beta}_b^*$ for simplicity.
- Correction won't catch external sources of bias; be warned.

Covariance:

- We may estimate the covariance as well:

$$\widehat{\text{Cov}}(\widehat{\boldsymbol{\beta}}) = \frac{1}{B-1} \sum_{b=1}^B (\widehat{\boldsymbol{\beta}}_b^* - \widehat{\boldsymbol{\beta}}^*)(\widehat{\boldsymbol{\beta}}_b^* - \widehat{\boldsymbol{\beta}}^*)^T$$

- For the k^{th} coefficient, we have:

$$\widehat{v}_k = \widehat{\text{Var}}(\widehat{\beta}_k) = \frac{1}{B} \sum_{b=1}^B ([\widehat{\boldsymbol{\beta}}_b^*]_k - \widehat{\beta}_k^*)^2$$

Confidence intervals: Normal approximation (bias-correction)

- Symmetric $(1 - \alpha)$ CI:

$$(\hat{\beta}_k - \widehat{\text{Bias}}_k) \pm \sqrt{\widehat{v}_k} z_{1-\alpha/2}$$

- Assumptions:

- ▶ $\hat{\beta}_k - \beta_k \sim \mathcal{N}(\text{Bias}_k, \sigma^2)$, which is symmetric and pivotal.
- ▶ $\widehat{\text{Bias}}_k$ and \widehat{v}_k are good estimates of Bias_k and σ^2 .

- Good for cases where N is large enough that normal approximation holds, but no known theoretical formula for asymptotic variance.
- Can use QQ-plots to evaluate departures from normality.

Confidence intervals: Pivot based

- Let $\hat{\beta}_{k(p)}^*$ denote p^{th} quantile of k^{th} coefficient of $\{\hat{\beta}_k^*\}_{b=1}^B$.
- Behavior of $\beta_k - \hat{\beta}_k$ approximately that of $\hat{\beta}_k - \hat{\beta}_k^*$:

$$\begin{aligned}
 0.95 &\approx \text{P}\left(\hat{\beta}_{k(\alpha/2)}^* \leq \hat{\beta}_k^* \leq \hat{\beta}_{k(1-\alpha/2)}^*\right) \\
 &= \text{P}\left(\hat{\beta}_k - \hat{\beta}_{k(1-\alpha/2)}^* \leq \hat{\beta}_k - \hat{\beta}_k^* \leq \hat{\beta}_k - \hat{\beta}_{k(\alpha/2)}^*\right) \\
 &\approx \text{P}\left(\hat{\beta}_k - \hat{\beta}_{k(1-\alpha/2)}^* \leq \beta_k - \hat{\beta}_k \leq \hat{\beta}_k - \hat{\beta}_{k(\alpha/2)}^*\right) \\
 &= \text{P}\left(2\hat{\beta}_k - \hat{\beta}_{k(1-\alpha/2)}^* \leq \beta_k \leq 2\hat{\beta}_k - \hat{\beta}_{k(\alpha/2)}^*\right)
 \end{aligned}$$

- Assumptions:
 - ▶ $\hat{\beta}_k - \beta_k$ asymptotically pivotal (not necessarily symmetric).

Confidence intervals:

- There are plenty of other of bootstrap-based confidence intervals. One simple one I did not cover is based on the quantiles of the bootstrap samples.
- The pivot-based confidence interval is generally understood to have better properties.
- See empirical process theory for all kinds of other generalizations, extensions, theoretical results.

Linear regression: Fixed design

- Re-sample residuals $\widehat{\epsilon}_i^*$ from the existing residuals $\{\widehat{\epsilon}_i\}_{i=1}^N$ with replacement.
- Keep \mathbf{x}_i intact and form N new outcomes as $y_i^* = \mathbf{x}_i^T \widehat{\boldsymbol{\beta}} + \widehat{\epsilon}_i^*$ for $i = 1, \dots, N$.
- Estimate $\widehat{\boldsymbol{\beta}}_b^*$ for $b = 1, \dots, N$; form estimates/confidence intervals of your choosing from prior methods.
- Assumptions:
 - ▶ Homoscedasticity of errors.
 - ▶ Correct mean-model.
- Consistent with a designed experiment/randomized trial.
- If \mathbf{X} is discrete, you can simply leave the \mathbf{x} 's as they are and resample the outcomes separately within subgroup of \mathbf{X} .

Linear regression: Random design

- Re-sample pairs (\mathbf{x}_i^*, y_i^*) from existing observations $\{\mathbf{x}_i, y_i\}_{i=1}^N$ with replacement.
- Estimate $\hat{\boldsymbol{\beta}}_b^*$ for $b = 1, \dots, N$; form estimates/confidence intervals of your choosing from prior methods.
- Design changes with each sample.
- Consistent with an observational study.

Linear regression: Fixed vs. random design

- Assume homoscedastic errors.
- If the mean model is correct, either version of the bootstrap should perform well regardless of whether \mathbf{X} is fixed by design or random.
- If \mathbf{X} is fixed by design, mean-model misspecification will tend to result in an overstated variance if you treat \mathbf{X} as random.
- If \mathbf{X} is random by design, mean-model misspecification will tend to result in an understated variance if you treat \mathbf{X} as fixed.

Stata: Example (MRI)

- `regress height age, robust (recall)`
- `regress height age, vce(bs, reps(500))`
- `regress height age, vce(bs, reps(500) nodots)`
- `estat bootstrap, all`

Stata: Example (MRI)

```
. regress height age, robust
```

```
Linear regression
```

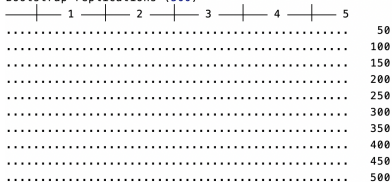
```
Number of obs   =      735
F(1, 733)       =      9.21
Prob > F        =     0.0025
R-squared       =     0.0120
Root MSE       =     9.6581
```

height	Robust		t	P> t	[95% Conf. Interval]	
	Coef.	Std. Err.				
age	-0.1953694	.0643711	-3.04	0.002	-0.3217432	-0.0689956
_cons	180.3453	4.805937	37.53	0.000	170.9103	189.7804

Stata: Example (MRI)

```
. regress height age, vce(bs, reps(500))
(running regress on estimation sample)
```

```
Bootstrap replications (500)
```



```
Linear regression                Number of obs   =       735
                                Replications      =       500
                                Wald chi2(1)        =       8.40
                                Prob > chi2        =     0.0038
                                R-squared           =     0.0120
                                Adj R-squared       =     0.0107
                                Root MSE        =     9.6581
```

height	Observed Coef.	Bootstrap Std. Err.	z	P> z	Normal-based [95% Conf. Interval]	
age	-.1953694	.0674101	-2.90	0.004	-.3274907	-.0632481
_cons	180.3453	5.000509	36.07	0.000	170.5445	190.1461

Stata: Example (MRI)

```
. regress height age, vce(bs, reps(500) nodots)
```

Linear regression

```
Number of obs   =      735
Replications    =      500
Wald chi2(1)    =       8.97
Prob > chi2     =     0.0027
R-squared       =     0.0120
Adj R-squared   =     0.0107
Root MSE       =     9.6581
```

height	Observed Coef.	Bootstrap Std. Err.	z	P> z	Normal-based [95% Conf. Interval]	
age	-.1953694	.0652377	-2.99	0.003	-.323233	-.0675058
_cons	180.3453	4.874817	37.00	0.000	170.7909	189.8998

Stata: Example (MRI)

```
. estat bootstrap, all
```

```
Linear regression                Number of obs    =       735
                                Replications      =       500
```

height	Observed Coef.	Bias	Bootstrap Std. Err.	[95% Conf. Interval]		
age	-.19536938	-.0014101	.06523773	-.323233	-.0675058	(N)
				-.3367485	-.0664426	(P)
				-.3296939	-.0654481	(BC)
_cons	180.34533	.1100677	4.8748171	170.7909	189.8998	(N)
				170.8138	190.7536	(P)
				170.6618	190.2488	(BC)

```
(N) normal confidence interval
(P) percentile confidence interval
(BC) bias-corrected confidence interval
```

Stata: Example (MRI)

- N: Normal CI
- P: Percentile CI
- BC: Bias-corrected CI

Notes: Topics in this unit

- Reminder of typical inference procedures.
- The bootstrap.
 - ▶ A powerful tool that allows you to conduct inference and form confidence intervals in settings where you may not be able to trust model-based or sandwich standard errors.
- There is plenty more to say about the bootstrap. Take advanced regression courses to learn more! :)

Notes: Next unit

- Bayesian methods!